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A New Option Jump-Diffusion Model: A Simple Formula [*]

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Abstract

In this paper, we provide an alternative to the jump-diffusion option pricing models. In doing so, we provide a simple, explicit formula that doesn't require a computational method. Furthermore, we introduce a new, simple method for solving partial integro-differential equation equations.

Keywords: Option pricing, Merton model, jump-diffusion, closed-form solution, partial integro- differential equation, the Black-Scholes formula.

JEL Classification: G0

1 Introduction

In response to some of the limitations of the Black-Scholes model, Merton (1976) introduced a seminal jump-diffusion model for the price of the European call option. However, his well-known and highly cited formula is not a closed form; it is an infinite sum (approximation) that requires a computational method. Furthermore, some of the parameters and probability distribution assumptions can be eliminated.

Later models on jump diffusions such as Kou (2002), Zhang and Wang (2013), Zhu et al (2013) and Gong and Zhuang (2016). require computational or numerical methods. Svishchuk et al (2000) made theoretical contributions under certain assumptions. In addition, the previous models do not clearly capture the intuitive and desirable features captured by the model we introduce in this paper.

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In this paper, we overcome these limitations. In doing so, without a loss of generality, we provide a far simpler, explicit formula that doesn't require any numerical/computational methods. Furthermore, our formula is perfectly intuitive. Also, our formula is a small modification of the Black-Scholes formula. Thus, it is also easily and directly comparable to the Black-Scholes formula.

Moreover, we introduce a new method for solving partial differential-difference equations. In doing so, we devise a simple method to transform a partial differential-difference equation to a partial differential equation.

The rest of the paper is organized as follows. Section 2 provides a brief description of the Merton model in the literature. Section 3 first develops the theorem and then provides a verification, and thereafter, provides an example for the theory we developed in this paper. Section 4 empirically compares the performance of the estimates obtained by using our formula to those using Merton's formula. Section 5 concludes, discusses the limitations of the theory we developed in our paper and suggests further extensions of the theory.

2 The Model

In this section, we provide a brief description of the Merton model in the literature. To do so, we first discuss the dynamics of the stock price. We then discuss the dynamics of the option price. Thereafter, we discuss Merton's partial integro-differential equation and the Merton pricing formula.

The Merton model, see Merton (1973, 1976), views corporate default as the event where the firm's asset value falls below debt at maturity mathematically equivalent to pricing equity as a call option. First, we present the stock price using the following formula:

$$S_t = \Pi_{j=1}^{n} Y_j S e^{(\alpha - \frac{\sigma^2}{2} - \lambda k)t + \sigma Z_t}, \tag{2.1}$$

where Z is a Gaussian variable, n is a Poisson Process, λt is its intensity,Y_j are identically and independently distributed, α is expected return rate, and σ is the volatility. Readers may read Merton (1973, 1976) for the definitions of other terms.

Stock price dynamics describe how stock prices evolve over time mathematically. In finance, this is usually modeled using stochastic processes. One can extend Merton (1973, 1976) to obtain the dynamics of the stock price as given by the following formula:

$$dS = S\left[(\alpha - \lambda k)\, dt + \sigma dZ + (Y_t - 1)\, dn_t\right], \tag{2.2}$$

where $S \equiv S_t$, $Y_t - 1$ is the relative jump size (independent of dn_t), and k is its mean. Readers may refer to Lucas and McDonald (1990) for the definitions of other terms.

Option prices move because the underlying stock price evolves stochastically. Their dynamics come from combining the stock price process, no-arbitrage pricing, and hedging arguments. The dynamics of the option price are given by the following formula:

$$dC\,(t, S) = C\,(t, S)\,((\alpha_c - \lambda k_c)\, dt + \sigma_c dZ + dq_c), \tag{2.3}$$

where

$$\alpha_c = \left(C_t + (\alpha - \lambda k)\, SC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} + \lambda E_Y\left[C\,(t, YS) - C\,(t, S)\right]\right) / C\,(t, S),$$

$$\sigma_c = \sigma SC_S / C\,(t, S),$$

$Y \equiv Y_t$, the subscripts of C are partial derivatives, and q_c is an independent Poisson process. Readers may refer to Chernov, *et al.* (2003) for the definitions of other terms.

Merton's partial integro-differential equation comes from the jump-diffusion model he developed in 1976, where stock prices follow continuous diffusion (Brownian motion) and sudden jumps (Poisson process). This extends the Black-Scholes framework developed by and later expanded by Black and Scholes (1972). Readers may refer to Matsuda (2004) for more information.

By using Merton's assumptions such as a diversifiable jump risk and risk neutrality, we obtain Merton's well-known partial integro-differential equation as shown in the following formula:

$$C_t + (r - \lambda k) SC_S - rC + \frac{1}{2}\sigma^2 S^2 C_{SS} + \lambda E_Y \left[C(t, YS) - C(t, S) \right] = 0, \tag{2.4}$$

where r is the interest rate (a constant), and YS is the price of the underlying asset after the jump. We note that Merton's partial integro-differential equation (PIDE) is obtained by extending the Black-Scholes model to allow for jumps in asset prices (the jump-diffusion model).

Applying the above equations, one could then easily obtain the Merton pricing formula (under the assumption of the log-normality of the jump size) as shown in the following formula:

$$C(t, S) = \sum_{i=0}^{\infty} e^{-\bar{\lambda}(T-t)} \frac{\left(\bar{\lambda}(T-t)\right)^i}{i!} C_{BS}(\sigma_i, r_i, S, T-t), \tag{2.5}$$

where C_{BS} is the Black-Scholes price, $\bar{\lambda} = (1+k)\lambda$, $r_i = r - \lambda k + \frac{i \ln(1+k)}{T-t}$, and $\sigma_i^2 = \sigma^2 + \frac{i\delta^2}{T-t}$, where $\delta^2 = Var(\ln Y)$ and $T - t$ is the time to expiry.

Based on the above equations, we are now ready to develop the model in the next section.

3 The New Model, Verification and Example

In this section, we first develop the theorem and then provide a verification, and thereafter, provide an example for the theory we developed in this paper.

3.1 The New Model

We first develop the theorem for the theory we developed in this paper. To do so, we first present the dynamics of the price of the underlying asset as shown in the following formula:

$$dS = S\left[rdt + \sigma dB + (Y - 1)\, dn_t \right], \tag{3.1}$$

where $B \sim N(-\lambda kt/\sigma, t)$ (see, for example, Matsuda (2004)).

By using Equation 2.5, one can obtain the Merton partial integro-differential equation (with a slight modification) as shown in the following formula:

$$C_t + r\left(SC_S - C\right) + \frac{1}{2}\sigma^2 S^2 C_{SS} + \lambda E_Y\left[C\left(t, YS\right) - C\left(t, S\right)\right] = 0. \tag{3.2}$$

Using the above equations, we obtain the following proposition:

Proposition 3.1 *The price of the option is*

$$C\left(0, S\right) = e^{\lambda \hat{\varphi}_i T} C_{BS} = e^{\lambda \hat{\varphi}_i T}\left[SN\left(d_1\right) - e^{-rT} KN\left(d_2\right)\right], \tag{3.3}$$

where λ *and* $\hat{\varphi}_i$ *are constants,* $d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ *and* $d_2 = d_1 - \sigma\sqrt{T}$.

Proof. Let $\frac{E_Y[C(t,YS) - C(t,S)]}{C(t,S)} = \varphi_t$, so that $E_Y\left[C\left(t, YS\right) - C\left(t, S\right)\right] = \varphi_t C\left(t, S\right)$. Thus, (3.2) can be given by (suppressing the notation)

$$C_t + r\left(SC_S - C\right) + \frac{1}{2}\sigma^2 S^2 C_{SS} + \lambda \varphi_t C\left(t, S\right) = 0. \tag{3.4}$$

Therefore,

$$C_t + rSC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} + \left(\lambda \varphi_t - r\right) C = 0, C\left(T, S\left(T\right)\right) = g\left(S\right), \tag{3.5}$$

where g is the payoff of the option. This is a generalized Black-Scholes partial differential equation. Conditioning on each value of φ (given $\varphi = \varphi_i$), its solution is (see, for example, Alghalith (2018)).[1]

$$\bar{C}\left(0, S, \varphi_i\right) = e^{\lambda \varphi_i T}\left[SN\left(d_1\right) - e^{-rT} KN\left(d_2\right)\right] = e^{\lambda \varphi_i T} C_{BS}, \tag{3.8}$$

where $d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$, K is the strike price, and C_{BS} is the Black-Scholes price.

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[1] The generalized Black-Scholes partial differential equation is

$$C_t + \theta SP_S + \frac{1}{2}\sigma^2 S^2 C_{SS} - \gamma C = 0, C\left(T, S\left(T\right)\right) = g\left(S\right). \tag{3.6}$$

Its solution is

$$e^{(\theta - \gamma)T} SN\left(d_1\right) - e^{-\gamma T} KN\left(d_2\right), \tag{3.7}$$

where $d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(S/K\right) + \left(\theta + \sigma^2/2\right)T\right]$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Now, the option price can be expressed as a weighted average of these prices conditional on φ as follows

$$C\left(0,S\right)=\int_{\varphi_i} e^{\lambda\varphi_i T}C_{BS}dF\left(\varphi_i\right), \tag{3.9}$$

where F is the cumulative density. By the continuity, the expected value is a specific value of $\bar{C}\left(\varphi_i\right)$ denoted by $\hat{C}\left(\varphi_i\right)=C\left(\hat{\varphi}_i\right)=e^{\lambda\hat{\varphi}_i T}C_{BS}$, where $\hat{\varphi}_i$ is a value (outcome) of φ.

Thus, the price of the option is

$$C\left(0,S\right)=e^{\lambda\hat{\varphi}_i T}C_{BS}=e^{\lambda\hat{\varphi}_i T}\left[SN\left(d_1\right)-e^{-rT}KN\left(d_2\right)\right], \tag{3.10}$$

where $d_1=\frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $d_2=d_1-\sigma\sqrt{T}$. ■

Similar to the parameters of the classical model, the parameter $\hat{\varphi}_i$ can be estimated using similar methods or other methods. In addition, the implied value of $\hat{\varphi}_i$ can be calculated and then used in the estimation of $\hat{\varphi}_i$.

3.2 Verification

In this section, we will provide a verification for the theory we developed in this paper.

A simple way to verify the result, let $C^*=E\left[e^{-r(T-t)}g\left(S_T\right)\right]$ is the true option price with jumps (by definition); by the continuity, there is a specific value of the parameter ϕ such as $\hat{\phi}$, such that $C^*=e^{\lambda\hat{\varphi}_i T}C_{BS}$.

3.3 Numerical Example

In this section, we will provide the following numerical example for the theory we developed in this paper:

Example 3.1 *If* $S=100$, $K=90$, $r=.05$, $T=1$, $\sigma=.25$, $\lambda=1$ *and* $\hat{\varphi}_i=.01$, *then the price of the European call* $C\left(0,S\right)=18.14e^{.01}=\$18.32$.

The finding in Example 3.1 shows that the theory we developed in this paper can be used to analyse numerical data.

4 Empirical Comparisons and Estimation

In this section, we empirically compare the performance of the estimates obtained by using our formula to those using Merton's formula. In doing so, we need to devise a method to estimate $\hat{\varphi}_i$. One way to estimate it is to use Taylor's expansion of $E_Y C\left(t, YS\right)$ around S as follows:

$$C\left(0, YS\right) \approx C_{BS}\left(S\right) + \varpi\left[YS - S\right],$$

thus,

$$E_Y C\left(0, YS\right) \approx C_{BS}\left(S\right) + \varpi Sk.$$

Therefore, by using Equation (3.4), $\hat{\varphi}_i$ can be estimated as

$$\hat{\varphi}_i \approx \frac{\varpi Sk}{C_{BS}\left(S\right)}.$$

Also, since the jump is a change in the stock price, ϖ can be estimated as the delta of the Black-Scholes price $\nabla = \frac{\partial C_{BS}(S)}{\partial S}$; thus, we have

$$\hat{\varphi}_i \approx \frac{\nabla Sk}{C_{BS}\left(S\right)}.$$

For example, by using the data in the first row of Table 1, we obtain $\nabla \approx 1$ and $\hat{\varphi}_i \approx .01238$. Thus, we have $C\left(0, S\right) = e^{.01238*2*.28} C_{BS} = e^{.01238*2*.28} * 4.26 = 4.29$.

In Table 1, we use the same data as Storeng (2014) (see the appendix in Storeng (2014) for more information on the data). We compute the price estimated by using the theory developed in this paper (we call it our price), the price estimated by using Merton's formula (we call it the Merton price), and the market price under different maturities and exhibit the results in Table 1.

Table 1. Empirical Results

	Market price	Merton price	Our price
$T = 103\ days, r = .24\%$	4.30	4.26	4.29
$T = 201\ days, r = .35\%$	4.40	4.29	4.35
$T = 257\ days, r = .42\%$	5.40	5.29	5.36

$S = 15.25, K = 11, \sigma = .20025, \lambda = 2, k = .003456$

By using the results shown in Table 1, we can compare the performances of our price, Merton's price, and the market price under different maturities. By doing this, we find that in each case, our price is closer to the market price than Merton's price. Thus, if the market price is the benchmark, then we can conclude that our price is more accurate than Merton's price in all the cases under our study.

5 Conclusion

In response to some of the limitations of the Black-Scholes model, Merton (1976) introduced a seminal jump-diffusion model for the price of the European call option. However, there are some limitations to his formula. For example, his formula does not have a closed form, and it is an infinite sum (approximation) that requires a complicated computational method. Furthermore, some of the parameters and probability distribution assumptions in his formula can be eliminated. There are some extensions of Mertons formula, see, for example, Kou (2002), Zhang and Wang (2013), Zhu et al (2013), Gong and Zhuang (2016), and Svishchuk et al (2000). However, there are still some limitations of the extensions. To circumvent the limitations of Mertons formula and its extensions, in this paper, we develop a formula for the price of the option that is far simpler, explicit formula that doesn't require any numerical/computational methods. We also develop a theorem with a proof to show the formula is valid. Furthermore, our formula is perfectly intuitive. Also, our formula is a small modification of the Black-Scholes formula. Thus, it is also easily and directly comparable to the Black-Scholes formula. We also provide a verification and a numerical example for the theory we developed in this paper. In addition, we empirically compare the performance of the estimates obtained by using our formula to those obtained using Merton's formula to show the superiority of our formula.

In sum, this result is perfectly intuitive since if there is no jump, $\lambda = 0$ and thus the price will be equal to the Black-Scholes price. Moreover, the option price increases in λ. Aside from the simplicity, it is intuitively very appealing. We also found out that our price is closer to the market price than Merton's price based on our data. This method can be applied to other models in finance or mathematics in the future. Our approach could be used to get better use of the option price in applications, see, for example, Wong, et al. (2010).

There are several limitations of using our approach, for example, by using our approach, theta needs to be approximated by using Taylor Expansion. Future extensions of our paper could include other stochastic volatility or stochastic interest rates in the equation.

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